1. Mean- Value Theorem (class on 28.12.2021) by G. Panigrahi

Use Mean Value Theorem to prove the following inequalities.

**Problem1: Prove that**  0< - <1

We know that 0<θ<1, so obviously - , this quantity is equals to ‘θ’ after certain expansion of some function f(x).

Now to prove this inequality consider f(x)= ,Now expand it in the interval [0,x] and obtain the value of θ.

We know from mean value theorem ,with Lagrange’s from:

f(b)=f(a)+h.f’(a+θh) as interval is [0,x] so ‘b’ is ‘x’ and ‘a’ is ‘0’ ,so we can write this theorem as

f(x)=f(0)+x.f’(θx) as h=(b-a) so h=(x-0)=x

so f(x)== 0+

or = or +θ= or θ= - (Proved)

**Problem2: Use Mean Value Theorem to prove the following inequalities.**

0< < 1

We know that 0<θ<1 ,so obviously 0< <1 , this quantity is equals to ‘θ’ after certain expansion of some function f(x).

Now to prove this inequality consider f(x)= ,Now expand it in the interval [0,x] and obtain the value of θ.

f(b)=f(a)+h.f’(a+θh) as interval is [0,x] so ‘b’ is ‘x’ and ‘a’ is ‘0’ ,so we can write that

f(x)=f(0)+x.f’(θx) as h=(b-a) so h=(x-0)=x

so f(x)== + x.

or 1+ x.=

or x.=-1

or = or θx= log or θ= log

So 0< log <1 [Proved]

**Generalised Mean Value Theorem.**

**Conditions :**

If a real valued function f of a real variable x is defined in [a,a+h] be such that

i)The (n-1)th derivative fn-1 is continuous in [a,a+h]

ii) The nth derivative fn exists in in the open interval (a,a+h)

then there exists at least one number θ ,where 0 <θ<1 such that

**f(a+h)=f(a) + h.f’(a) + f’’(a) +………………………+ fn(a+θh)**

**Proof:**

As (n-1)th derivative is continuous then f,f’ ,f’’ ……….,fn-2 are continuous in the closed interval [a,b]

Let us define a continuous function using the continuous functions

**Ø(x) = f(x) + (a+h-x).f’(x) + f’’(x) +…………………………………+ . f(n-1) + (a+h-x)nA**

Observation from the above function

Other than f(x) all other terms are multiplied with certain powers of (a+h-x) . so if we put x=a+h other than f(x) all the terms in right hand side becomes 0 and ø(a+h)=f(a+h)

And above in the expansion of f(a+h) is given so if there is a freedom to supply the quantity of A ,any value ,then we can assume that ø(a)=ø(a+h).

If in the function ø ,if we put x=a then it is just like the expansion of f(a+h) ,only the last term is different ,the last term becomes .A.

So if we can prove A is equal to , we can able to prove the generalised mean value theorem with Lagrange’s form of remainder ..

Now i)ø(x) is continuous

ii) ø(x) is derivable

iii)ø(a)=ø(a+h) by assumption

then ø(x) satisfies all the conditions of Rolle’s theorem.

So ø’(x)=0 at a certain value in the interval [a,a+h] , so it can be written as ø’(a+θh))=0 where 0<θ<1

Observation : if we calculate ø’(x) all the terms will be cancelled ,only the last two terms remains.

ø’(x)= f’(x)- f’(x) +(a+h-x)f’’(x) – (a+h-x)f’’(x) +……………..+ f(n-1) – n(a+h-x)n-1 . A

as olny last two terms remains so last two terms equated to 0

so f n  – n(a+h-x)n-1 . A=0 Now A= (Proved)

if **ø(x)= f(x)+(a+h-x)f’(x) +………………………….+fn-1(x) +(a+h-x)A**

Where A is taken so that ø(x)=ø(a+h)

So Rolle’s theorem can be applied

Ø’(x)=0 or fn(x) -1.A=0

Now x= a+θh so put the value of x in the numerator we get

**=**A or  **Now h. =**

**Maclaurin’s theorem:**

In case of the interval [0,x] instead of [a,b] the corresponding result will be named after Maclaurin. Thus Maclaurin’s theorem with Lagrange’s form of remainder can be written as:

**f(x)= f(0)+xf’(0)+ f’’(x) +………………………..+fn(θx)**

Thus Maclaurin’s theorem with Cauchy’s form of remainder can be written as

**f(x)= f(0)+xf’(0)+ f’’(x) +………………………..+fn(θx)**

**Problem3:**

Apply mean value theorem to the appropriate order to prove that

**x> log(1+x)> x- for x>0**

f(x)= log(1+x) f(0)=0

f’(x)= f’(0)=1

f’’(θx)=-

log(1+x)= 0+ x.1 - . Now a quantity is subtracted from x ,where denominator is 2. which is greater than 2. So when denominator is larger total quantity is less. So a less quantity is subtracted from x.

So x.1 - . > x.1 - . And x> x.1 - . because no quantity is subtracted from x.

**Problem:4** **if f(h)=f(0) +h.f’(0) +f’’(θh) , 0<θ<1**

Then prove that θ= when h=1

And f(x)= Now f(0)=1

f’(x)= .-1 now f’(0)=.-1

f’’(x)= . Now f’’(θh)=.

f(1)=0,

so 0= 1- + .. or . = or = ..=

or so θ=

**Problem5: Obtain the expansion of the following functions**

**In the interval [0,x]**

f(x)=

=1+ - + -

f(0)=1

f’(x)=

f’’(x)= .- Now f’’(0)= -x 2

f’’’(x)= .-. (1+x) Now . f’’’(x)= ..-.=

so f’’’(0)=

Now f4(x)= .-..

Now ..-.. = -

So f(x)==1+ - + - [Proved]